Efficient Discretization of Stochastic Integrals

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Abstract

Sharp asymptotic lower bounds of the expected quadratic variation of discretization error in stochastic integration are given. The theory relies on inequalities for the kurtosis and skewness of a general random variable which are themselves seemingly new. Asymptotically efficient schemes which attain the lower bounds are constructed explicitly. The result is directly applicable to practical hedging problem in mathematical finance; it gives an asymptotically optimal way to choose rebalancing dates and portofolios with respect to transaction costs. The asymptotically efficient strategies in fact reflect the structure of transaction costs. In particular a specific biased rebalancing scheme is shown to be superior to unbiased schemes if transaction costs follow a convex model. The problem is discussed also in terms of the exponential utility maximization.

1 Introduction

The stochastic integral $X \cdot Y_\sigma$ with respect to a semimartingale Y and a stopping time σ is by definition a limit of $X^n \cdot Y_\sigma$ in probability, where X^n is a sequence of simple predictable processes with $\sup_{t\in[0,\sigma]} |X^n_t - X_t| \to 0$ in probability as $n \to \infty$. This convergence of $X^n \cdot Y$ is essential not only for the theoretical construction of the stochastic integral but also for practical approximations in problems modeled with stochastic integrals. The aim of this paper is to give a way to choose X^n efficiently in an asymptotic sense. The main assumption of the paper is that X is a continuous semimartingale.

Denote by K the Radon-Nikodym derivative of the absolutely continuous part of $\langle Y\rangle$ with respect to $\langle X\rangle$, which always exists in light of the Lebesgue decomposition theorem. Fukasawa [4] showed that

$$\liminf_{n\to\infty} \mathbb{E}[N[X^n]_\sigma]\mathbb{E}[\langle Z[X^n]\rangle_\sigma] \geq \frac{1}{6}\mathbb{E}[\sqrt{K} \cdot \langle X\rangle_\sigma],$$

where $N[X^n]_\sigma$ is the number of the jumps of a given simple predictable process X^n up to σ and $Z[X^n] := (X - X^n) \cdot Y$ is the associated approximation error. If Y is a local martingale, then $\mathbb{E}[|Z[X^n]_\sigma|^2] = \mathbb{E}[\langle Z[X^n]\rangle_\sigma]$ under a reasonable assumption, and so the above inequality gives an asymptotic lower bound

of the mean squared error of discretization. Notice that the bound does not depend on X^n. The inequality is sharp in that the lower bound is attained by

$$\begin{aligned} &X^n_t := X_{\tau^n_j}, \ \ t \in (\tau^n_j, \tau^n_{j+1}], \ \ j = 0, 1, \dots, \\ &\tau^n_0 := 0, \ \ \tau^n_{j+1} := \inf\{t > \tau^n_j; |X_t - X_{\tau^n_j}| = \epsilon_n K^{-1/4}_{\tau^n_j}\}, \ \ \epsilon_n \downarrow 0 \end{aligned} \tag{1}$$

under a reasonable condition. We call such a sequence X^n that attains the lower bound an asymptotically efficient scheme. The above result is extended and proved under a less restrictive condition in this paper as a particular case.

To obtain a precise approximation to $X \cdot Y$, one has to take X^n as close to X as possible. In practical contexts it may be inevitably accompanied by various kinds of cost, especially if X is not of finite variation. The number of jumps $N[X^n]_\sigma$ is interpreted as one of them. In the context of mathematical finance for example, X and Y stand for a portfolio strategy and an asset price process respectively. Then $Z[X^n]$ represents the replication error associated to a discrete rebalancing strategy X^n. A continuous rebalancing is impossible in practice and $N[X^n]_\sigma$ corresponds to the number of trading, a measure on trader's effort. The scheme (1) defines an asymptotically efficient discrete strategy which asymptotically minimizes the mean squared error relative to the specific cost function $\mathbb{E}[N[X^n]_\sigma]$.

The sequence $\mathbb{E}[N[X^n]_\sigma]$ is however just one of measures on costs. Again for example in the financial context, the cumulative transaction cost associated to X^n is often modeled as

$$\kappa \sum_{0<t\leq\sigma} Y_t |\Delta X^n_t|$$

with a constant $\kappa > 0$. This is the so-called linear or proportional transaction cost model. More generally one may consider as a cost or penalty,

$$C[S, \beta; X^n] := \sum_{0<t\leq\sigma} S_t K_t |\Delta X^n_t|^\beta 1_{\{|\Delta X^n|>0\}} \tag{2}$$

with a nonnegative predictable process S and a constant $\beta \geq 0$. Notice that $C[1/K, 0; X^n]_\sigma$ and $C[Y/K, 1; X^n]_\sigma$ represent the number of rebalancing and the cumulative linear transaction cost respectively. If $\beta \in (0, 1)$ or $\beta > 1$, the cost is concave or convex respectively in the amount of transaction. Beyond these interpretations in the financial context, we treat the general form of $C[S, \beta; X^n]_\sigma$ as a penalty against taking X^n too close to X. Then a natural problem would be to minimize $\mathbb{E}[\langle Z[X^n]\rangle_\sigma]$ relative to the expected cost $\mathbb{E}[C[S, \beta; X^n]_\sigma]$ in the asymptotic situation that $\sup_{t\in[0,\sigma]} |X^n_t - X_t| \to 0$. Fukasawa [2](in Japanese) proposed this framework and proved that for all $\beta \in [0, 2)$,

$$\liminf_{n\to\infty} |\mathbb{E}[C[S, \beta; X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}[\langle Z[X^n]\rangle_\sigma] \geq \frac{1}{6} |\mathbb{E}[(S^{2/(4-\beta)} K) \cdot \langle X\rangle_\sigma]|^{(4-\beta)/(2-\beta)} \tag{3}$$

if X^n is of the form $X^n_t = X_{\tau^n_j}$ for any $t \in (\tau^n_j, \tau^n_{j+1}]$ with an increasing sequence of stopping times $\tau^n = \{\tau^n_j\}$ with $\tau^n_0 = 0$ and $\sup_{j\geq 0} |\tau^n_{j+1} \wedge \sigma - \tau^n_j \wedge \sigma| \to 0$ as $n \to \infty$.

The lower bound is sharp in that it is attained by

$$\begin{aligned} &X^n_t := X_{\tau^n_j}, \quad t \in (\tau^n_j, \tau^n_{j+1}], \quad j = 0, 1, \dots, \\ &\tau^n_0 = 0, \quad \tau^n_{j+1} = \inf\left\{ t > \tau^n_j ; |X_t - X_{\tau^n_j}| \geq \epsilon_n S^{1/(4-\beta)}_{\tau^n_j} \right\}, \quad \epsilon_n \downarrow 0 \end{aligned} \tag{4}$$

under a reasonable condition. The proof is given in this paper as well under a less restrictive condition. This result does not give a complete answer to our problem in that the lower bound is for a restricted class of X^n as $X^n_t = X_{\tau^n_j}$ for $t \in (\tau^n_j, \tau^n_{j+1}]$ with some $\{\tau^n_j\}$. We call such X^n an unbiased scheme. Intuitively, taking X^n in the unbiased manner is natural and necessary to have a good approximation to $X \cdot Y$. In fact in the case $\beta = 0$ and $C[S, \beta, X; X^n] = N[X^n]_\sigma$, as stated first, the unbiased scheme X^n defined by (1) is asymptotically efficient. The main result of this paper shows that the discretization scheme (4) is actually asymptotically efficient if $\beta \in [0, 1]$, however not so if $\beta \in (1, 2)$. In the latter case, surprisingly, the lower bound is reduced to one third and asymptotically attained by a sequence of biased schemes.

In Section 2, we give a general result on the centered moments of a random variable, which seems new and important itself and plays an essential role to derive lower bounds of discretization error in the stochastic integration. In Section 3, we give a sharp lower bound for unbiased schemes, which is a slight extension of the result of Fukasawa [2](in Japanese). In Section 4, we give sharp lower bounds for possibly biased schemes and construct explicit schemes which asymptotically attain the bounds. In Section 5, we show that an asymptotically efficient scheme is a maximizer of a scaling limit of the exponential utility in the financial context of discrete hedging.

We conclude this section by mentioning related studies in the literature. Rootzén [15] studied the discretization error of stochastic integrals with the equidistant partition $\tau^n_j = j/n$ and proved that the discretization error of a stochastic integral converges in law to a time-changed Brownian motion with rate $n^{-1/2}$ as $n \to \infty$. An extension to discontinuous semimartingales was given by Tankov and Voltchkova [16] in the equidistant case. Fukasawa [3] gave an extension to another direction that admits a general sequence of locally homogeneous stochastic partitions and gave several sharp lower bounds of the asymptotic conditional variance of the discretization error. Hayashi and Mykland [10] revisited Rootzén's problem in terms of the discrete hedging in mathematical finance. Motivated by this financial application, the mean squared error was studied by Gobet and Temam [9], Geiss and Geiss [6], Geiss and Toivola [7] under the Black-Scholes model. Among others, Geiss and Geiss [6] showed that the use of stochastic partitions does not improve the convergence rate. In a sense our result refines this observation under a general framework. Our problem is also related to Leland's strategy for hedging under transaction costs. See Leland [14], Denis and Kabanov [1], Fukasawa [5]. The difference is that we are looking for an efficient discrete hedging strategy which does not require a surcharge, while Leland's strategy does it to absorb transaction costs. In a statistical framework, Genon-Catalot and Jacod [8] studied an opti-

mality problem for a class of random sampling schemes, which is smaller than our class. Finally remark that the use of hitting times such as (4) has another advantage in terms of almost sure convergence. See Karandikar [12].

2 Kurtosis-skewness inequalities

Here we study the centered moments of a general random variable. The reason why we need such a general framework is that in our problem of discretization, we encounter the moments of a martingale evaluated at a stopping time, which can follow any distribution with mean 0 in light of Skorokhod stopping problem. The notation in this section is independent of that in other sections. We say a random variable X is Bernoulli if the support of X consists of two points. We say X is symmetrically Bernoulli if X is Bernoulli and its skewness is 0, that is, $\mathbb{E}[(X-\mathbb{E}[X])^3] = 0$. For any random variable X with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] > 0$ and $\mathbb{E}[X^4] < \infty$, it holds that

$$\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3} \geq 1. \tag{5}$$

This is often called Pearson's inequality and easily shown as follows:

$$|\mathbb{E}[X^3]|^2 = |\mathbb{E}[X(X^2 - \mathbb{E}[X^2])]|^2 \leq \mathbb{E}[X^2](\mathbb{E}[X^4] - |\mathbb{E}[X^2]|^2).$$

From this proof it is clear that the equality is attained only if X is Bernoulli. Conversely if X is Bernoulli, then we get the equality by a straightforward calculation. Pearson's inequality was used by Fukasawa [3][4] to obtain lower bounds of discretization error of stochastic integrals. This is however not sufficient for our current purpose. Fukasawa [3] proved another inequality which looks similar to but independent of (5):

$$\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \frac{3}{4}\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3} \geq \frac{\mathbb{E}[X^2]}{|\mathbb{E}[|X|]|^2}. \tag{6}$$

The equality is attained if and only if X is Bernoulli. The proof is lengthy and unexpectedly different from that for Pearson's inequality. See Appendix B of Fukasawa [3]. From these inequalities we obtain the following lemmas.

Lemma 1 *Let $\beta \in [0, 1)$. For any random variable X with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] > 0$ and $\mathbb{E}[X^4] < \infty$,*

$$\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \frac{3}{4}\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3} \geq \frac{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}}. \tag{7}$$

The equality is attained if and only if X is symmetrically Bernoulli.

Proof: By Hölder's inequality, we have

$$\mathbb{E}[|X|] \leq |\mathbb{E}[X^2]|^{(1-\beta)/(2-\beta)}|\mathbb{E}[|X|^\beta]|^{1/(2-\beta)},$$

or equivalently,

$$\frac{\mathbb{E}[X^2]}{|\mathbb{E}[|X|]|^2} \geq \frac{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}}.$$

The result then follows from (6). ////

Lemma 2 *Let $\beta \in [0, 2)$ and $\alpha \in [0, 1]$. For any random variable X with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] > 0$ and $\mathbb{E}[X^4] < \infty$,*

$$\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \alpha\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3} - (1-\alpha)\frac{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}} \geq \alpha. \tag{8}$$

The equality is attained if and only if X is symmetrically Bernoulli.

Proof: By Hölder's inequality, we have

$$\mathbb{E}[X^2] \leq |\mathbb{E}[|X|^\beta]|^{2/(4-\beta)}|\mathbb{E}[X^4]|^{(2-\beta)/(4-\beta)},$$

or equivalently,

$$\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} \geq \frac{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}}.$$

Therefore,

$$\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \alpha\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3} - (1-\alpha)\frac{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}} \geq \alpha\left\{\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3}\right\}.$$

The result then follows from (5). ////

Lemma 3 *Let $\beta \in [0, 2)$ and $\alpha \in (0, 1]$. For any random variable X with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] > 0$ and $\mathbb{E}[X^4] < \infty$,*

$$\frac{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}}{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}\left\{\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \alpha\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3}\right\} > 1 - \alpha. \tag{9}$$

Moreover if X is Bernoulli, then

$$\frac{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}}{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}\left\{\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \alpha\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3}\right\} = F_{\alpha\beta}\left(\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3}\right), \tag{10}$$

where $F_{\alpha\beta}$ is a continuous function with $F_{\alpha\beta}(0) = 1$. If $\beta \in (1, 2)$, then $F_{\alpha\beta}(\infty) = 1 - \alpha$.

Proof: The inequality (9) is apparent from (5) and (8). Let X be Bernoulli. We suppose $\mathbb{E}[X^2] = 1$ without loss of generality. Then the support of X is of the form $\{e^x, -e^{-x}\}$ and $\mathbb{P}[X = e^x] = 1/(1 + e^{2x})$ with $x \in \mathbb{R}$. By a straightforward calculation, we get $\mathbb{E}[X^3] = 2\sinh(x)$ and

$$\frac{|\mathbb{E}[|X|^\beta]|^{2/(2-\beta)}}{|\mathbb{E}[X^2]|^{\beta/(2-\beta)}}\left\{\frac{\mathbb{E}[X^4]}{|\mathbb{E}[X^2]|^2} - \alpha\frac{|\mathbb{E}[X^3]|^2}{|\mathbb{E}[X^2]|^3}\right\} = \frac{4\alpha - 3 + 4(1-\alpha)|\cosh(x)|^2}{|\cosh(x)|^{2/(2-\beta)}|\cosh((\beta-1)x)|^{-2/(2-\beta)}}.$$

Putting

$$g(x) = \cosh((\beta-1)x)|\cosh(x)|^{1-\beta}, \tag{11}$$

the right hand side is given by

$$\frac{4\alpha-3}{g(x)^{-2/(2-\beta)}|\cosh(x)|^2} + \frac{4(1-\alpha)}{g(x)^{-2/(2-\beta)}}. \tag{12}$$

Notice that $g(0) = 1$ and $g(x)^{-2/(2-\beta)}$ converges to 4 as $|x| \to \infty$ for $\beta \in (1,2)$. ////

Remark 4 Let g be defined by (11). Since

$$g'(x) = (\beta-1)\sinh((\beta-2)x)|\cosh(x)|^{-\beta}, \quad g''(0) = (1-\beta)(2-\beta),$$

for $\beta \neq 1$, $g'(x) = 0$ if and only if $x = 0$. Further if $\beta \in [0,1)$ or $\beta \in (1,2)$, respectively, the minimum or maximum of g is attained at $x = 0$. Therefore if $\alpha \geq 3/4$ and $\beta \in (1,2)$, the function defined by (12) is decreasing in $|x|$ and converges to $1-\alpha$ as $|x| \to \infty$. However in the following sections, we use Lemma 3 with $\alpha = 2/3$, where the function is not necessarily monotone in $|x|$.

3 Efficiency for unbiased Riemann sums

Here we recall the problem with a rigorous formulation and give a slight improvement of the result of Fukasawa [2]. Let X and Y be semimartingales defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ which satisfies the usual conditions. We assume that that there exist a continuous local martingale M and a locally bounded adapted process H such that

$$X = H \cdot \langle M \rangle + M.$$

Denote by $\mathcal{T}$ the set of the increasing sequences of stopping times $\tau = \{\tau_j\}$ with $0 = \tau_0 < \tau_1 < \cdots$ and $\lim_{j\to\infty} \tau_j = \infty$ a.s.. Given $\tau = \{\tau_j\} \in \mathcal{T}$, define a simple predictable process $X[\tau]$ as $X[\tau]_t = X_{\tau_j}$ for $t \in (\tau_j, \tau_{j+1}]$. Conversely, for a given simple predictable process $\hat{X}$, define $\tau[\hat{X}] \in \mathcal{T}$ as the sequence of the jump times of $\hat{X}$. By definition we have

$$Z[X[\tau]]_t = \int_0^t X_s \mathrm{d}Y_s - \sum_{j=0}^{\infty} X_{\tau_j}(Y_{\tau_{j+1}\wedge t} - Y_{\tau_j \wedge t}),$$
$$Z[\hat{X}]_t = \int_0^t X_s \mathrm{d}Y_s - \sum_{j=0}^{\infty} \hat{X}_{\tau[\hat{X}]_j+}(Y_{\tau[\hat{X}]_{j+1}\wedge t} - Y_{\tau[\hat{X}]_j \wedge t})$$

for $t \geq 0$. Our aim is to minimize $\mathbb{E}[\langle Z[X^n]\rangle_\sigma]$ asymptotically when

$$\sup_{t\in[0,\sigma]} |X^n_t - X_t| \to 0 \tag{13}$$

in probability as $n \to \infty$. Denote by K the Radon-Nikodym derivative of the absolutely continuous part of the predictable quadratic variation $\langle Y \rangle$ with respect to $\langle X \rangle$, which always exists in light of the Lebesgue decomposition theorem. We consider the cost $C[S, \beta; \hat{X}]$ defined by (2) for a given simple predictable process $\hat{X}$. We assume that K and S are positive, continuous and moreover, constant on any random interval where $\langle X \rangle$ is constant. By the last assumption, we have

$$K = \tilde{K}_{\langle X \rangle}, \quad \tilde{K} = K_F, \quad S = \tilde{S}_{\langle X \rangle}, \quad \tilde{S} = S_F, \tag{14}$$

where $F_s = \inf\{t \geq 0; \langle X \rangle_t > s\}$; see Karatzas and Shreve [13], 3.4.5.

Now we define a class of unbiased schemes in which at first we consider the efficiency or optimality of discretization. Denote by $\mathcal{T}_u(S, \beta, \sigma)$ the set of the sequences of simple predictable processes X^n of the form $X^n = X[\tau^n]$, $\tau^n = \{\tau^n_j\} \in \mathcal{T}$, such that there exists a sequence of stopping times σ^m with $\sigma^m \to \sigma$ as $m \to \infty$,

1. for each m, (13) holds with σ^m instead of σ.
2. for each m,
$$\mathbb{E}[C[S, \beta; X^n]_{\sigma^m}]^{2/(2-\beta)} \langle Z[X^n] \rangle_{\sigma^m}$$
is uniformly integrable in n.

Remark 5 The uniform integrability condition for $\mathcal{T}_u(S, \beta, \sigma)$ is usually easy to check. It is for example satisfied when considering the sequence of the equidistant partitions $\tau^n_j = j/n$ if $\mathrm{d}\langle X \rangle_t$ has a locally bounded Radon-Nikodym derivative with respect to $\mathrm{d}t$. The exponent $2/(2-\beta)$ is actually chosen so that $\mathbb{E}[C[S, \beta; X^n]_{\sigma^m}]^{2/(2-\beta)} \propto n$ asymptotically in the equidistant case since n^{-1} is the optimal convergence rate of $\langle Z[X^n] \rangle_{\sigma^m}$ for the case. All reasonable X^n should enjoy this property of rate-efficiency. Note that by the Dunford-Petis theorem, the uniform integrability is equivalent to the relative compactness in the $\sigma(L^1, L^\infty)$ topology. By the Eberlein-Smulian theorem, it is further equivalent to the relative sequential compactness in the same topology.

Theorem 6 *Let $\beta \in [0, 2)$. The inequality (3) holds for all $\{X^n\} \in \mathcal{T}_u(S, \beta, \sigma)$.*

For the proof, we start with a lemma.

Lemma 7 *Let X^n be a sequence of simple predictable processes. Then (13) implies that*

$$\sup_{j \geq 0} |\langle X \rangle_{\tau^n_{j+1} \wedge \sigma} - \langle X \rangle_{\tau^n_j \wedge \sigma}| \to 0 \tag{15}$$

in probability as $n \to \infty$ with $\tau^n = \tau[X^n]$. Conversely if (15) holds for a sequence $\tau^n \in \mathcal{T}$, then (13) holds with $X^n = X[\tau^n]$.

Proof: For any subsequence of n, there exists a further subsequence n_k such that (13) holds a.s. with $n = n_k$ as $k \to \infty$. It suffices then to show that (15) holds

a.s. with this subsequence. Let Ω^* be a subset of Ω such that for any $\omega \in \Omega^*$, (15) does not hold with $n = n_k$, $k \to \infty$. Then, for $\omega \in \Omega^*$, there exist $\epsilon(\omega) > 0$ and a sequence of intervals $I_m(\omega) = [a_m(\omega), b_m(\omega)]$ such that for each m, there exists $n = n_k$ such that $I_m(\omega) = [\tau^n_j(\omega), \tau^n_{j+1}(\omega)]$ and

$$\inf_m |\langle X\rangle_{b_m}(\omega) - \langle X\rangle_{a_m}(\omega)| \geq \epsilon(\omega).$$

Since $(a_m(\omega), b_m(\omega))$ is a sequence in the compact set $[0,\sigma(\omega)] \times [0,\sigma(\omega)]$, it has an accumulating point $[a_*(\omega), b_*(\omega)]$ with

$$|\langle X\rangle_{b_*}(\omega) - \langle X\rangle_{a_*}(\omega)| \geq \epsilon(\omega).$$

With probability one, $\langle X\rangle$ is continuous, so we may suppose that $a_*(\omega) < b^*(\omega)$ without loss of generality. Again with probability one, if X is constant on an interval, then $\langle X\rangle$ is constant on the interval. So we may suppose that $X(\omega)$ is not constant on $[a_*(\omega), b_*(\omega)]$ without loss of generality. On the other hand, there exists a subsequence $X^m(\omega)$ of $X^{n_k}(\omega)$ such that $X^m(\omega)$ is constant on a non-empty interval of $[a_*(\omega), b_*(\omega)]$. Recalling the way that the subsequence was chosen, we conclude that $\mathbb{P}[\Omega^*] = 0$. ////

Proof of Theorem 6: Put $\tau^n = \tau[X^n]$. By the usual localization argument, we may and do suppose without loss of generality that $X, \langle X\rangle, K, 1/K, S$ and H are bounded up to σ, that (15) holds, and that $|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\langle Z[X^n]\rangle_\sigma$ is uniformly integrable in n. Define $K[\tau]$ as $K[\tau]_t = K_{\tau_j}$ for $t \in [\tau_j, \tau_{j+1})$ for $\tau \in \mathcal{T}$. Let

$$\epsilon^n = \sup_{0\leq s\leq\sigma} |K_s - K[\tau^n]_s|.$$

By Lemma 7 and (14), we have that ϵ_n is bounded and converges to 0 in probability as $n \to \infty$. By Itô's formula,

$$\begin{aligned}
\langle Z[X^n]\rangle_t &= \int_0^t (X_s - X^n_s)^2 \mathrm{d}\langle Y\rangle_s \\
&\geq \int_0^t (X_s - X^n_s)^2 K_s \mathrm{d}\langle X\rangle_s \\
&= \int_0^t (X_s - X^n_s)^2 K[\tau^n]_s \mathrm{d}\langle X\rangle_s + \int_0^t (X_s - X^n_s)^2 (K_s - K[\tau^n]_s)\mathrm{d}\langle X\rangle_s \qquad (16)\\
&= \frac{1}{6}\sum_{j=0}^\infty K_{\tau^n_j}(X_{\tau^n_{j+1}\wedge t} - X_{\tau^n_j \wedge t})^4 - \frac{2}{3}\int_0^t K[\tau^n]_s (X_s - X^n_s)^3 \mathrm{d}X_s \\
&\quad + \int_0^t (X_s - X^n_s)^2 (K_s - K[\tau^n]_s)\mathrm{d}\langle X\rangle_s.
\end{aligned}$$

Now we show that

$$\lim_{n\to\infty} \mathbb{E}\left[|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \int_0^\sigma (X_s - X^n_s)^2 (K_s - K[\tau^n]_s)\mathrm{d}\langle X\rangle_s\right] = 0.$$

Put

$$V^n = |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \langle Z[X^n]\rangle_\sigma$$
$$= |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \int_0^\sigma (X_s - X^n_s)^2 \mathrm{d}\langle Y\rangle_s.$$

Since $1/K$ is bounded by a constant, say, $A > 0$ and $K_s \mathrm{d}\langle X\rangle_s \leq \mathrm{d}\langle Y\rangle_s$, we have

$$|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \int_0^\sigma (X_s - X^n_s)^2 |K_s - K[\tau^n]_s| \mathrm{d}\langle X\rangle_s \leq A\epsilon^n V^n \to 0$$

in probability. Since ϵ^n is bounded and V^n is uniformly integrable, $\epsilon^n V^n$ is uniformly integrable as well and so, we obtain that $\mathbb{E}[\epsilon^n V^n] \to 0$.

Similarly, we can show that

$$|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}\left[\int_0^\sigma K[\tau^n]_s (X_s - X^n_s)^3 \mathrm{d}X_s\right]$$
$$= |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}\left[\int_0^\sigma K[\tau^n]_s (X_s - X^n_s)^3 H_s \mathrm{d}\langle X\rangle_s\right] \to 0$$

by using the continuity of X instead of K. So far we have obtained

$$\liminf_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}[\langle Z[X^n]\rangle_\sigma]$$
$$\geq \liminf_{n\to\infty} \frac{1}{6} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}\left[\sum_{j=0}^\infty K_{\tau^n_j} (X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma})^4\right].$$

On the other hand, by Hölder's inequality,

$$\mathbb{E}\left[\sum_{j\geq 1, \tau^n_j \leq \sigma} |S_{\tau^n_j}|^{2/(4-\beta)} |K_{\tau^n_{j-1}}|^{1/p} |K_{\tau^n_j}|^{1/q} (X_{\tau^n_j} - X_{\tau^n_{j-1}})^2\right]$$
$$\leq \left|\mathbb{E}\left[\sum_{j=0}^\infty K_{\tau^n_j} (X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma})^4\right]\right|^{1/p} \left|\mathbb{E}\left[\sum_{0<t\leq\sigma} S_t K_t |\Delta X^n_t|^\beta 1_{\{|\Delta X^n_t|>0\}}\right]\right|^{1/q}$$
$$= \left|\mathbb{E}\left[\sum_{j=0}^\infty K_{\tau^n_j} (X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma})^4\right]\right|^{1/p} \left|\mathbb{E}\left[C[S,\beta,X^n]_\sigma\right]\right|^{1/q}$$

where $p = (4-\beta)/(2-\beta)$ and $q = p/(p-1) = (4-\beta)/2$. The left hand side converges to $\mathbb{E}[(S^{2/(4-\beta)}K) \cdot \langle X\rangle_\sigma]$. ////

Theorem 8 *Suppose that $\langle Y\rangle = K \cdot \langle X\rangle$. Let $\hat{S}$ be a positive continuous adapted process which is constant on any random interval where $\langle X\rangle$ is constant. Let ϵ_n be a positive sequence with $\epsilon_n \to 0$ as $n \to \infty$. Define X^n as*

$$X^n_t := X_{\tau^n_j}, \quad t \in [\tau^n_j, \tau^n_{j+1}), \quad j = 0, 1, \ldots,$$
$$\tau^n_0 := 0, \quad \tau^n_{j+1} := \inf\{t > \tau^n_j; |X_t - X_{\tau^n_j}| = \epsilon_n \hat{S}_{\tau^n_j}\}. \tag{17}$$

Then $\{X^n\} \in \mathcal{T}_u(S,\beta,\sigma)$ *for any* $\beta \in [0,2)$*. Moreover if* $X, \langle X\rangle, H, K, 1/K, S, 1/S, \hat{S}$ *and* $1/\hat{S}$ *are bounded up to* σ*, then we have that for any* $\beta \in [0,2)$*,*

$$\sum_{j=0}^{\infty} |X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}|^2, \quad \frac{C[S,\beta;X^n]_\sigma}{\mathbb{E}[C[S,\beta;X^n]_\sigma]}$$

are uniformly integrable in n*, and*

$$\lim_{n\to\infty} \epsilon_n^{2-\beta} \mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} = \mathbb{E}[(S\hat{S}^{\beta-2}) \cdot \langle Y\rangle_\sigma]$$
$$\lim_{n\to\infty} \epsilon_n^{-2} \mathbb{E}[\langle Z[X^n]\rangle_\sigma] = \frac{1}{6}\mathbb{E}[\hat{S}^2 \cdot \langle Y\rangle_\sigma].$$

In particular if $\hat{S} = S^{1/(4-\beta)}$*, or equivalently,* X^n *is defined by (4), then*

$$\lim_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}[\langle Z[X^n]\rangle_\sigma] = \frac{1}{6}|\mathbb{E}[(S^{2/(4-\beta)}K)\cdot\langle X\rangle_\sigma]|^{(4-\beta)/(2-\beta)}.$$

Proof: By the usual localization argument, we may and do suppose without loss of generality that $X, \langle X\rangle, H, K, 1/K, S, 1/S, \hat{S}$ and $1/\hat{S}$ are bounded up to σ. Then, notice that the uniformly integrability of

$$\sum_{j=0}^{\infty} |X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}|^2 \tag{18}$$

follows from the decomposition

$$\sum_{j=0}^{\infty} |X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}|^2 = \langle X\rangle_\sigma + 2\int_0^\sigma (X_t - X^n_t)H_t \mathrm{d}\langle X\rangle_t + 2\int_0^\sigma (X_t - X^n_t)\mathrm{d}M_t.$$

Let us show $X^n \in \mathcal{T}_u(S,\beta,\sigma)$. The convergence (13) is apparent by definition. Since

$$\begin{aligned} C[S,\beta;X^n]_\sigma &= \sum_{0<t\leq\sigma} S_t K_t |\Delta X^n_t|^{\beta-2} |\Delta X^n_t|^2 1_{\{|\Delta X^n_t|>0\}} \\ &= \epsilon_n^{\beta-2} \sum_{j\geq 1, \tau^n_j\leq\sigma} S_{\tau^n_j} K_{\tau^n_j} \hat{S}^{\beta-2}_{\tau^n_{j-1}} |X_{\tau^n_j} - X_{\tau^n_{j-1}}|^2, \end{aligned} \tag{19}$$

there exists a constant $c > 0$ such that

$$\frac{1}{c}\mathbb{E}\left[\sum_{j=0}^{\infty} |X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}|^2\right] \leq \epsilon_n^{2-\beta}\mathbb{E}[C[S,\beta;X^n]_\sigma] \leq c\mathbb{E}\left[\sum_{j=0}^{\infty} |X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}|^2\right].$$

Since

$$\mathbb{E}\left[\sum_{j=0}^{\infty} |X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}|^2\right] = \mathbb{E}[\langle X\rangle_\sigma] + \mathbb{E}[\int_0^\sigma (X_s - X^n_s)H_s \mathrm{d}\langle X\rangle_s],$$

we obtain $\mathbb{E}[C[S,\beta;X^n]_\sigma] = O(\epsilon_n^{\beta-2})$. On the other hand,

$$\langle Z[X^n]\rangle_\sigma \leq \sup_{t\in[0,\sigma]}\{K_t|X_t - X^n_t|^2\}\langle X\rangle_\sigma \leq \epsilon_n^2\langle X\rangle_\sigma \sup_{t\in[0,\sigma]}\hat{S}_t \sup_{t\in[0,\sigma]} K_t,$$

and so, we conclude that

$$|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\langle Z[X^n]\rangle_\sigma \tag{20}$$

is uniformly integrable. So far we showed that $X^n \in \mathcal{T}_u(S,\beta,\sigma)$. The uniform integrability of

$$\frac{C[S,\beta;X^n]_\sigma}{\mathbb{E}[C[S,\beta;X^n]_\sigma]}$$

also follows from that of (18) in light of (19). With the aid of the uniform integrability of (18) and (20), repeating the same argument as in the proof of Theorem 6, the convergence result follows from the fact that

$$\begin{aligned}
\epsilon_n^{2-\beta}C[S,\beta;X^n]_\sigma &= \sum_{j\geq 1,\tau^n_j\leq\sigma} S_{\tau^n_j}K_{\tau^n_j}\hat{S}^{\beta-2}_{\tau^n_{j-1}}|X_{\tau^n_j} - X_{\tau^n_{j-1}}|^2 \\
&\to \int_0^\sigma S_t\hat{S}_t^{\beta-2}K_t\mathrm{d}\langle X\rangle_t = (S\hat{S}^{\beta-2})\cdot\langle Y\rangle_\sigma, \\
\epsilon_n^{-2}\sum_{j\geq 1,\tau^n_j\leq\sigma} K_{\tau^n_{j-1}}(X_{\tau^n_j} - X_{\tau^n_{j-1}})^4 &= \sum_{j\geq 1,\tau^n_j\leq\sigma} K_{\tau^n_{j-1}}\hat{S}^2_{\tau^n_{j-1}}|X_{\tau^n_j} - X_{\tau^n_{j-1}}|^2 \\
&\to \int_0^\sigma \hat{S}_t^2K_t\mathrm{d}\langle X\rangle_t = \hat{S}^2\cdot\langle Y\rangle_\sigma
\end{aligned}$$

in probability as $n\to\infty$. ////

Remark 9 The assumption $\langle Y\rangle = K\cdot\langle X\rangle$ implies in particular that Y is quasi-left-continuous. That Y is quasi-left-continuous is equivalent to that Y has no predictable jump time. See Jacod and Shiryaev [11] for more details. For example, the Lévy processes are quasi-left-continuous. Of course so are the continuous semimartingales. The asymptotic efficiency of (4) is no more true if Y is not quasi-left continuous. In fact, if there is a predictable time τ such that $Y_\tau \neq Y_{\tau-}$, it is apparently more efficient to include τ, or more precisely, a time immediately before τ into the sequence of stopping times for discretization. This is possible because τ is predictable.

4 Efficiency for possibly biased Riemann sums

4.1 The case of $\beta\in[0,1]$

The class $\mathcal{T}_u(S,\beta,\sigma)$ was a set of unbiased schemes, that is, $\{X^n\}$ of the form $X^n = X[\tau^n]$, $\tau^n\in\mathcal{T}$. As an approximating sequence X^n to X, we may consider more general simple predictable processes. In this section we answer the question

that the scheme (4) is asymptotically efficient in a more general class of simple predictable processes or not. First we get a positive answer for $\beta \in [0,1]$. The result improves Fukasawa [4] for the case $\beta = 0$. Denote by $\mathcal{T}(S,0,\sigma)$ the set of the sequences X^n of simple predictable processes such that that there exists a sequence of stopping times σ^m with $\sigma^m \to \infty$ as $m \to \infty$,

1. for each m,
$$\sup_{t\in[0,\sigma^m]} |X^n_t - X_t|$$
is uniformly bounded and converges to 0 in probability as $n \to \infty$, and

2. for each m,
$$\mathbb{E}[C[S,0;X^n]_{\sigma^m}]\langle Z[X^n]\rangle_{\sigma^m}$$
is uniformly integrable in n.

For $\beta \in (0,2)$, we need additional conditions from technical point of view. We define $\mathcal{T}(S,\beta,\sigma)$ for $\beta \in (0,2)$ as the set of the sequences X^n of simple predictable processes such that there exists a sequence of stopping times σ^m with $\sigma^m \to \infty$ as $m \to \infty$,

1. for each m,
$$\sup_{t\in[0,\sigma^m]} |X^n_t - X_t|, \quad \sup_{t\in[0,\sigma^m]} \left|\frac{\Delta X^n_t}{\Delta X[\tau[X^n]]_t} - 1\right|$$
are uniformly bounded and converge to 0 in probability as $n \to \infty$, where $0/0$ is understood as 1, and

2. for each m,
$$|\mathbb{E}[C[S,\beta;X^n]_{\sigma^m}]|^{2/(2-\beta)}\langle Z[X^n]\rangle_{\sigma^m}, \quad \frac{C[S,\beta;X[\tau[X^n]]]_{\sigma^m}}{\mathbb{E}[C[S,\beta;X[\tau[X^n]]]_{\sigma^m}]}$$
are uniformly integrable in n.

The convergence of the ratio between $\Delta X[\tau[X^n]]$ and ΔX^n to 1 means that X^n cannot be too biased. Of course it always holds if X^n is unbiased since $X[\tau[X^n]] = X^n$. The uniform integrability of the normalized cost function associated with $X[\tau[X^n]]$ is reasonable in that it requires the sequence of stopping times $\tau[X^n]$ to be sufficiently regular. By Theorem 8, the scheme $\{X^n\}$ defined by (17) is an element of $\mathcal{T}(S,\beta,\sigma)$ for any $\beta \in [0,2)$. Therefore, the following theorem asserts that the scheme $\{X^n\}$ defined by (4) is asymptotically efficient in the class $\mathcal{T}(S,\beta,\sigma)$ if $\beta \in [0,1]$.

Theorem 10 *Let $\beta \in [0,1]$. The inequality (3) holds for all $\{X^n\} \in \mathcal{T}(S,\beta,\sigma)$.*

Proof: Write $\tau^n = \tau[X^n]$ for brevity. By the usual localization procedure, we may and do suppose without loss of generality that $X, H\cdot M, \langle X\rangle, K, 1/K, S, 1/S$ and H are bounded up to σ, that $\sup_{t\in[0,\sigma]}|X^n_t - X_t|$ is uniformly bounded and converge

to 0, and that $|\mathbb{E}[C[S,\beta;X[\tau[X^n]]]_\sigma]|^{2/(2-\beta)}\langle Z[X^n]\rangle_\sigma$ is uniformly integrable in n. For the case $\beta \in (0,1]$, we may have additionally that

$$\sup_{t\in[0,\sigma]}\left|\frac{\Delta X^n_t}{\Delta X[\tau^n]_t}-1\right|$$

is uniformly bounded and converge to 0, and that

$$\frac{C[S,\beta;X[\tau^n]]_\sigma}{\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]}$$

is uniformly integrable in n. Define K and $K[\tau^n]$ as in the proof of Theorem 6. By Itô's formula,

$$\begin{aligned}\langle Z[X^n]\rangle_\sigma \geq &\int_0^\sigma |X_s - X^n_s|^2 K[\tau^n]_s \mathrm{d}\langle X\rangle_s + \int_0^\sigma |X_s - X^n_s|^2 (K_s - K[\tau^n]_s)\mathrm{d}\langle X\rangle_s \\ =&\frac{1}{6}\sum_{j=0}^\infty K_{\tau^n_j}((\Delta_j+\delta_j)^4-\delta_j^4) - \frac{2}{3}\int_0^\sigma K[\tau^n]_s (X_s - X^n_s)^3 \mathrm{d}X_s \\ &+ \int_0^\sigma |X_s - X^n_s|^2 (K_s - K[\tau^n]_s)\mathrm{d}\langle X\rangle_s,\end{aligned}$$

where $\Delta_j = X_{\tau^n_{j+1}\wedge\sigma} - X_{\tau^n_j\wedge\sigma}$ and $\delta_j = X_{\tau^n_j\wedge\sigma} - X^n_{\tau^n_j\wedge\sigma}$. As before, we can show that

$$\begin{aligned}&\lim_{n\to\infty}|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\int_0^\sigma |X_s - X^n_s|^2(K_s - K[\tau^n]_s)\mathrm{d}\langle X\rangle_s] = 0, \\ &\lim_{n\to\infty}|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\int_0^\sigma K[\tau^n]_s(X_s - X^n_s)^3\mathrm{d}X_s] = 0\end{aligned}$$

by the uniform integrability with the aid of Lemma 7. Put

$$F_t = \exp\left\{\int_0^t H_s \mathrm{d}M_s - \frac{1}{2}\int_0^t H_s^2 \mathrm{d}\langle M\rangle_s\right\}.$$

Since

$$\mathbb{E}[F_{\tau^n_{j+1}}/F_{\tau^n_j}] = 1, \quad \sup_{t\geq 0, j\geq 0}\left|1 - \frac{F_{t\wedge\tau^n_{j+1}\wedge\sigma}}{F_{\tau^n_j\wedge\sigma}}\right| \to 0$$

in probability, again by Itô's formula, we have that

$$\begin{aligned}&\liminf_{n\to\infty}|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\langle Z[X^n]\rangle_\sigma] \\ &= \frac{1}{6}\liminf_{n\to\infty}|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}\left[\sum_{j=0}^\infty K_{\tau^n_j}((\Delta_j+\delta_j)^4-\delta_j^4)\right] \\ &= \frac{1}{6}\liminf_{n\to\infty}|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}\left[\sum_{j=0}^\infty K_{\tau^n_j}((\Delta_j+\delta_j)^4-\delta_j^4)\frac{F_{\tau^n_{j+1}\wedge\sigma}}{F_{\tau^n_j\wedge\sigma}}\right] \\ &= \frac{1}{6}\liminf_{n\to\infty}|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}\left[\sum_{j=0}^\infty K_{\tau^n_j}\hat{\mathbb{E}}_j[((\Delta_j+\delta_j)^4-\delta_j^4)]\right],\end{aligned}$$

where $\hat{\mathbb{E}}_j[A]$ refers to the conditional expectation $\mathbb{E}[AF_{\tau^n_{j+1}\wedge\sigma}/F_{\tau^n_j\wedge\sigma}|\mathcal{F}_{\tau^n_j\wedge\sigma}]$ for a random variable A. Notice that under $\hat{\mathbb{E}}_j$, $X_{t\wedge\tau^n_{j+1}\wedge\sigma} - X_{t\wedge\tau^n_j\wedge\sigma}$ is a martingale. Therefore,

$$\begin{aligned}\hat{\mathbb{E}}_j[((\Delta_j+\delta_j)^4-\delta_j^4)] =&\hat{\mathbb{E}}_j[\Delta_j^4]+4\delta_j\hat{\mathbb{E}}_j[\Delta_j^3]+6\delta_j^2\hat{\mathbb{E}}_j[\Delta_j^2]\\ =&6\hat{\mathbb{E}}_j[\Delta_j^2]\left(\delta_j+\frac{1}{3}\frac{\hat{\mathbb{E}}_j[\Delta_j^3]}{\hat{\mathbb{E}}_j[\Delta_j^2]}\right)^2+\hat{\mathbb{E}}_j[\Delta_j^4]-\frac{2}{3}\frac{|\hat{\mathbb{E}}_j[\Delta_j^3]|^2}{\hat{\mathbb{E}}_j[\Delta_j^2]}\\ \geq&\frac{|\hat{\mathbb{E}}_j[\Delta_j^2]|^{(4-\beta)/(2-\beta)}}{|\hat{\mathbb{E}}_j[|\Delta_j|^\beta]|^{2/(2-\beta)}}.\end{aligned}$$

Here, we have used Lemma 1 for $\beta\in[0,1)$ and (6) for $\beta=1$.

By Hölder's inequality,

$$\begin{aligned}&\mathbb{E}\left[\sum_{\tau^n_{j+1}\leq\sigma}^{\infty}|S_{\tau^n_j}|^{2/(4-\beta)}K_{\tau^n_j}\hat{\mathbb{E}}_j[\Delta_j^2]\right]\\ &\leq\left|\mathbb{E}\left[\sum_{j=0}^{\infty}\frac{K_{\tau^n_j}|\hat{\mathbb{E}}_j[\Delta_j^2]|^p}{|\hat{\mathbb{E}}_j[|\Delta_j|^\beta]|^{2/(2-\beta)}}\right]\right|^{1/p}\left|\mathbb{E}\left[\sum_{\tau^n_{j+1}\leq\sigma}S_{\tau^n_j}K_{\tau^n_j}\hat{\mathbb{E}}_j[|\Delta_j|^\beta]\right]\right|^{1/q},\end{aligned}$$

where $p=(4-\beta)/(2-\beta)$ and $q=p/(p-1)=(4-\beta)/2$. Since

$$\sup_{t\in[0,\sigma]}\left|\frac{|\Delta X^n_t|^\beta}{|\Delta X[\tau^n]_t|^\beta}-1\right|,\quad\sup_{\tau^n_{j+1}\leq\sigma}\left|\frac{S_{\tau^n_j}K_{\tau^n_j}}{S_{\tau^n_{j+1}}K_{\tau^n_{j+1}}}\frac{F_{\tau^n_{j+1}}}{F_{\tau^n_j}}-1\right|$$

are uniformly bounded and converge to 0 in probability, we get

$$\lim_{n\to\infty}\frac{\mathbb{E}\left[\sum_{\tau^n_{j+1}\leq\sigma}S_{\tau^n_j}K_{\tau^n_j}\hat{\mathbb{E}}_j[|\Delta_j|^\beta]\right]}{\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]}=\lim_{n\to\infty}\frac{\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]}{\mathbb{E}[C[S,\beta;X^n]_\sigma]}=1.$$

Here we have used the uniform integrability of $C[S,\beta;X[\tau^n]]_\sigma/\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]$ for $\beta\in(0,1]$. This is trivial if $\beta=0$.

By the bounded convergence theorem,

$$\lim_{n\to\infty}\mathbb{E}\left[\sum_{\tau^n_{j+1}\leq\sigma}|S_{\tau^n_j}|^{2/(4-\beta)}K_{\tau^n_j}\hat{\mathbb{E}}_j[\Delta_j^2]\right]=\mathbb{E}[(S^{2/(4-\beta)}K)\cdot\langle X\rangle_\sigma],$$

which completes the proof. ////

4.2 The case of $\beta\in(1,2)$

Here we show that the unbiased scheme X^n defined by (4) is no more efficient for $\beta\in(1,2)$. We give a lower bound which is one third the previous one and construct a biased scheme which asymptotically attains it.

Theorem 11 *Let $\beta \in (1,2)$. For all $\{X^n\} \in \mathcal{T}(S,\beta,\sigma)$,*

$$\liminf_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\langle Z[X^n]\rangle_\sigma] \geq \frac{1}{18}|\mathbb{E}[(S^{2/(4-\beta)}K)\cdot\langle X\rangle_\sigma]|^{(4-\beta)/(2-\beta)}.$$

Proof: Just use Lemma 3 with $\alpha = 2/3$ instead of Lemma 1 in the proof of Theorem 10. The rest is the same. ////

Theorem 12 *Suppose that $\langle Y\rangle = K\cdot\langle X\rangle$. Let $\beta \in (1,2)$ and ϵ_n be a positive sequence with $\epsilon_n \to 0$ as $n \to \infty$. For $\gamma \in \mathbb{R}$, define $\tau^n(\gamma) = \{\tau^n_j(\gamma)\}$ as*

$$\begin{aligned}
&\tau^n_0(\gamma) = 0,\ \ \tau^n_{j+1}(\gamma) = \min\{\tau^n_{j+1}(\gamma,+),\tau^n_{j+1}(\gamma,-)\},\\
&\tau^n_{j+1}(\gamma,+) = \inf\left\{t > \tau^n_j(\gamma); X_t - X_{\tau^n_j(\gamma)} \geq \epsilon_n e^{\gamma} S^{1/(4-\beta)}_{\tau^n_j(\gamma)}\right\},\\
&\tau^n_{j+1}(\gamma,-) = \inf\left\{t > \tau^n_j(\gamma); X_t - X_{\tau^n_j(\gamma)} \leq \epsilon_n e^{-\gamma} S^{1/(4-\beta)}_{\tau^n_j(\gamma)}\right\}.
\end{aligned} \tag{21}$$

Define a sequence of simple predictable processes $X^n(\gamma)$ as

$$X^n(\gamma) = X[\tau^n(\gamma)] + \frac{2}{3}\epsilon_n \sinh(\gamma) S[\tau^n(\gamma)]^{1/(4-\beta)}, \tag{22}$$

where $S[\tau^n(\gamma)]_t = S_{\tau^n_j(\gamma)}$ for $t \in [\tau^n_j(\gamma),\tau^n_{j+1}(\gamma))$. Then $\{X^n(\gamma)\} \in \mathcal{T}(S,\beta,\sigma)$. Moreover if $X,\langle X\rangle, H\cdot M, H, K, 1/K, S$ and $1/S$ are bounded up to σ, then

$$\lim_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n(\gamma)]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\langle Z[X^n(\gamma)]\rangle_\sigma] = \frac{F(|\gamma|)}{6}|\mathbb{E}[S^{2/(4-\beta)}\cdot\langle Y\rangle_\sigma]|^{(4-\beta)/(2-\beta)},$$

where F is a continuous function with $F(0) = 1$ and $F(\infty) = 1/3$. More explicitly,

$$F(x) = F(x,\beta) = \frac{4|\cosh(x)|^2 - 1}{3|\cosh(x)|^{2/(2-\beta)}|\cosh((\beta-1)x)|^{-2/(2-\beta)}}.$$

Proof: By the usual localization procedure, we may and do suppose without loss of generality that $X, H\cdot M, \langle X\rangle, K, 1/K, S, 1/S$ and H are bounded up to σ. Put $X^n = X^n(\gamma)$ and $\tau^n = \tau[X^n] = \tau^n(\gamma)$ for brevity. Then it follows from definition that

$$\sup_{t\in[0,\sigma]}|X^n_t - X_t|,\ \ \sup_{t\in[0,\sigma]}\left|\frac{\Delta X^n_t}{\Delta X[\tau^n]_t} - 1\right|$$

are uniformly bounded and converge to 0. By the same argument as in the proof of Theorem 8, we have that

$$|\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]|^{2/(2-\beta)}\langle Z[X^n]\rangle_\sigma,\ \ \frac{C[S,\beta;X[\tau^n]]_\sigma}{\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]}$$

are uniformly integrable in n. Since these imply in particular that

$$\lim_{n\to\infty}\frac{\mathbb{E}[C[S,\beta;X^n]_\sigma]}{\mathbb{E}[C[S,\beta;X[\tau^n]]_\sigma]} = 1,$$

we conclude $\{X^n\} \in \mathcal{T}(S, \beta, \sigma)$.

Let $\Delta_j = X_{\tau^n_{j+1}} - X_{\tau^n_j}$ and $\delta_j = X_{\tau^n_j} - X^n_{\tau^n_j}$. Then we obtain, in a similar manner to the proof of Theorem 10, that

$$\begin{aligned}
&\lim_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}[\langle Z[X^n]\rangle_\sigma] \\
&= \frac{1}{6} \liminf_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)} \mathbb{E}\left[\sum_{\tau^n_j \le \sigma} K_{\tau^n_j} \hat{\mathbb{E}}_j[((\Delta_j + \delta_j)^4 - \delta_j^4)]\right]
\end{aligned}$$

and that

$$\hat{\mathbb{E}}_j[((\Delta_j + \delta_j)^4 - \delta_j^4)] = 6\hat{\mathbb{E}}_j[\Delta_j^2]\left(\delta_j + \frac{1}{3}\frac{\hat{\mathbb{E}}_j[\Delta_j^3]}{\hat{\mathbb{E}}_j[\Delta_j^2]}\right)^2 + \hat{\mathbb{E}}_j[\Delta_j^4] - \frac{2}{3}\frac{|\hat{\mathbb{E}}_j[\Delta_j^3]|^2}{\hat{\mathbb{E}}_j[\Delta_j^2]},$$

where $\hat{\mathbb{E}}_j[A]$ refers to the conditional expectation $\mathbb{E}[AF_{\tau^n_{j+1}}/F_{\tau^n_j}|\mathcal{F}_{\tau^n_j}]$ for a random variable A. By the optional sampling theorem,

$$\hat{\mathbb{E}}_j[I\{\Delta_j = \epsilon_n e^{\gamma} S_{\tau^n_j}^{1/(4-\beta)}\}] = \frac{e^{-\gamma}}{e^{\gamma} + e^{-\gamma}}, \quad \hat{\mathbb{E}}_j[I\{\Delta_j = -\epsilon_n e^{-\gamma} S_{\tau^n_j}^{1/(4-\beta)}\}] = \frac{e^{\gamma}}{e^{\gamma} + e^{-\gamma}}$$

and so,

$$\begin{aligned}
&\hat{\mathbb{E}}_j[\Delta_j] = 0, \quad \hat{\mathbb{E}}_j[\Delta_j^2] = \epsilon_n^2 S_{\tau^n_j}^{2/(4-\beta)}, \quad \hat{\mathbb{E}}_j[\Delta_j^3] = 2\epsilon_n^3 \sinh(\gamma) S_{\tau^n_j}^{3/(4-\beta)}, \\
&\hat{\mathbb{E}}_j[|\Delta_j|^\beta] = \epsilon_n^\beta \frac{\cosh((\beta-1)\gamma)}{\cosh(\gamma)} S_{\tau^n_j}^{\beta/(4-\beta)}.
\end{aligned}$$

Moreover by Lemma 3,

$$\begin{aligned}
&\hat{\mathbb{E}}_j[\Delta_j^4] - \frac{2}{3}\frac{|\hat{\mathbb{E}}_j[\Delta_j^3]|^2}{\hat{\mathbb{E}}_j[\Delta_j^2]} \\
&= F(|\gamma|)\frac{|\hat{\mathbb{E}}_j[\Delta_j^2]|^{(4-\beta)/(2-\beta)}}{|\hat{\mathbb{E}}_j[|\Delta_j|^\beta]|^{2/(2-\beta)}} \\
&= F(|\gamma|)\left|\frac{\cosh((\beta-1)\gamma)}{\cosh(\gamma)}\right|^{-2/(2-\beta)} \epsilon_n^2 S_{\tau^n_j}^{2/(4-\beta)} \hat{\mathbb{E}}_j[\Delta_j^2]
\end{aligned}$$

with $F = F(\cdot, \beta)$, which satisfies $F(|\gamma|) \to 1/3$ as $|\gamma| \to \infty$. By definition of X^n, we have

$$\delta_j + \frac{1}{3}\frac{\hat{\mathbb{E}}_j[\Delta_j^3]}{\hat{\mathbb{E}}_j[\Delta_j^2]} = 0.$$

Therefore,

$$\lim_{n\to\infty} \epsilon_n^{-2} \mathbb{E}[\langle Z[X^n]\rangle_\sigma] = \frac{1}{6} F(|\gamma|)\left|\frac{\cosh((\beta-1)\gamma)}{\cosh(\gamma)}\right|^{-2/(2-\beta)} \mathbb{E}[S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma].$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty} \epsilon_n^{2-\beta}\mathbb{E}[C[S,\beta;X[\tau^n]_\sigma]\\
&= \lim_{n\to\infty} \mathbb{E}\left[\sum_{\tau_j^n\leq\sigma} S_{\tau_j^n}K_{\tau_j^n}\hat{\mathbb{E}}_j[|\Delta_j|^\beta]\right]\\
&= \frac{\cosh((\beta-1)\gamma)}{\cosh(\gamma)}\lim_{n\to\infty}\mathbb{E}\left[\sum_{\tau_j^n\leq\sigma} S_{\tau_j^n}^{2/(4-\beta)}K_{\tau_j^n}\hat{\mathbb{E}}_j[\Delta_j^2]\right]\\
&= \frac{\cosh((\beta-1)\gamma)}{\cosh(\gamma)}\mathbb{E}[S^{2/(4-\beta)}\cdot\langle Y\rangle_\sigma].
\end{aligned}$$

These convergences give the result. ////

Remark 13 The use of hitting times is essential to have a good performance. In fact if we consider a class of simple predictable processes X^n such that $\tau[X^n]_{j+1} - \tau[X^n]_j$ is $\mathcal{F}_{\tau[X^n]_j}$-measurable for each $j \geq 0$, then we can show that

$$\liminf_{n\to\infty} |\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\langle Z[X^n]\rangle_\sigma] \geq \frac{1}{2}|\mathbb{E}[(S^{2/(4-\beta)}K)\cdot\langle X\rangle_\sigma]|^{(4-\beta)/(2-\beta)}$$

when, for example, $X = Y$ and it is a Brownian motion. This is because the kurtosis $\hat{\mathbb{E}}_j[\Delta_j^4]|\hat{\mathbb{E}}_j[\Delta_j^2]|^{-2}$ and skewness $\hat{\mathbb{E}}_j[\Delta_j^3]|\hat{\mathbb{E}}_j[\Delta_j^2]|^{-3/2}$ of a conditionally standard normal random variable Δ_j are 3 and 0 respectively, while the lower bound of kurtosis is 1 attained by Bernoulli random variables. The above measurability condition was supposed in Genon-Catalot and Jacod [8].

5 Exponential utility maximization

The schemes $X^n = X[\tau^n]$ with (4) and $X^n = X^n(\gamma)$ defined by (22) with (21) are efficient for $\beta \in [0,1]$ and $\beta \in (1,2)$ respectively in that they attain the asymptotic lower bound of

$$|\mathbb{E}[C[S,\beta;X^n]_\sigma]|^{2/(2-\beta)}\mathbb{E}[\langle Z[X^n]\rangle_\sigma]$$

for a reasonable class of approximating simple predictable processes X^n. In the financial context of discrete hedging, we may interpret the cost function $C[S,\beta;\hat{X}]_\sigma$ as the cumulative transaction cost associated to the rebalancing scheme $\hat{X}$. If we do so, then a more natural criterion for the optimality of $\hat{X}$ should be given in terms of the expected utility of the terminal wealth $-Z[\hat{X}]_\sigma - C[S,\beta;\hat{X}]_\sigma$. In this section, we see that the efficient schemes maximize a scaling limit of the exponential utility

$$1 - \mathbb{E}[\exp\{-\alpha_n(-Z[X^n]_\sigma - C[S^n,\beta;X^n]_\sigma)\}],\ \ S^n = \kappa_n S,\ \ \alpha_n \to \infty,\ \ \alpha_n\kappa_n \to 0.$$

Here κ_n is a deterministic sequence, which we interpret as the coefficient of the transaction costs. Letting $\kappa_n \to 0$, we try to obtain an asymptotic but explicit

solution for the maximization problem which can be expected to have a good performance when κ_n is sufficiently small. If $\kappa_n \to 0$, then we can make both $\langle Z[X^n]\rangle_\sigma$ and $C[S^n, \beta; X^n]_\sigma$ converge to 0 by taking any $\{X^n\} \in \mathcal{T}(S, \beta, \sigma)$ such that $\sup_{t\in[0,\sigma]} |X^n_t - X_t| \to 0$ sufficiently slow. To find effective X^n among others, we consider a scaling limit by letting α_n, the risk-aversion parameter, diverge. In this section we assume Y to be continuous in addition. By Jacod's theorem of stable convergence of semimartingales, if there exists a continuous process V such that

$$\alpha_n^2\langle Z[X^n]\rangle_t \to V_t, \quad \alpha_n\langle Z[X^n], Y\rangle_t \to 0 \tag{23}$$

in probability for all $t \geq 0$, then $\alpha_n Z[X^n]$ converges $\mathcal{F}$-stably in law to a time-changed Brownian motion W_V, where W is a standard Brownian motion which is independent of $\mathcal{F}$. See Fukasawa [3] for more details and sufficient conditions for (23). Note that the second condition of (23) is to make the replication error $Z[X^n]$ asymptotically neutral to the market return. If in addition $\alpha_n C[S^n, \beta; X^n]_\sigma$ converges to a random variable C_σ in probability, then

$$\alpha_n Z[X^n]_\sigma + \alpha_n C[S^n, \beta; X^n]_\sigma \to W_{V_\sigma} + C_\sigma$$

in law. The limit law is a mixed normal distribution with conditional mean C_σ and conditional variance V_σ. This implies in particular that

$$1 - \mathbb{E}[\exp\{-\alpha_n(-Z[X^n]_\sigma - C[S^n, \beta; X^n]_\sigma)\}] \to 1 - \mathbb{E}[\exp\{C_\sigma + \frac{1}{2}V_\sigma\}]$$

under the uniform integrability condition on $\exp\{\alpha_n(Z[X^n]_\sigma + C[S^n, \beta; X^n]_\sigma)\}$. Then the maximization of the exponential utility reduces to the minimization of $C_\sigma + V_\sigma/2$. Under the additional assumptions that

$$\alpha_n^4 \sum_{j=0}^{\infty} \mathbb{E}[|\langle X\rangle_{\tau^n_{j+1}\wedge\sigma} - \langle X\rangle_{\tau^n_j\wedge\sigma}|^4 | \mathcal{F}_{\tau^n_j\wedge\sigma}] \to 0$$

in probability with $\tau^n = \tau[X^n]$ and that

$$\alpha_n^{(6-2\beta)/(2-\beta)}\kappa_n^{2/(2-\beta)} \to \mu > 0,$$

we obtain that

$$C_\sigma^{2/(2-\beta)} V_\sigma \geq \frac{\mu}{6}|S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma|^{(4-\beta)/(2-\beta)}$$

for $\beta \in [0,1]$ by a similar argument to the proof of Theorem 10 with the aid of Lemma A.2 of Fukasawa [3]. This is in fact an extension of Theorems 2.7 and 2.8 of Fukasawa [3]. It follows then that

$$\begin{aligned} C_\sigma + \frac{1}{2}V_\sigma &\geq C_\sigma + \frac{\mu}{12}|S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma|^{(4-\beta)/(2-\beta)} C_\sigma^{-2/(2-\beta)} \\ &\geq \hat{\mu} S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma, \end{aligned}$$

where

$$\hat{\mu} = \left|\frac{\mu}{6(2-\beta)}\right|^{(2-\beta)/(4-\beta)} + \frac{\mu}{12}\left|\frac{\mu}{6(2-\beta)}\right|^{-2/(4-\beta)}.$$

Here we have used the fact that for given $c > 0$, $\min_{x>0}\{x + cx^{-2/(2-\beta)}\}$ is attained at $x = (2c/(2-\beta))^{(2-\beta)/(4-\beta)}$. Therefore,

$$\lim_{n\to\infty}\{1 - \mathbb{E}[\exp\{-\alpha_n(-Z[X^n]_\sigma - C[S^n,\beta;X^n]_\sigma)\}]\} \leq 1 - \mathbb{E}[\exp\{\hat{\mu}S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma\}].$$

The upper bound is attained by the efficient scheme X^n defined by (4) with $\epsilon_n = \nu\alpha_n^{-1}$ and

$$\nu = \mu^{1/2}\left|\frac{\mu}{6(2-\beta)}\right|^{-1/(4-\beta)}.$$

This can be proved by applying Theorem 2.6 of Fukasawa [3]. For $\beta \in (1,2)$, similarly we get

$$C_\sigma^{2/(2-\beta)}V_\sigma \geq \frac{\mu}{18}|S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma|^{(4-\beta)/(2-\beta)}$$

and so,

$$\begin{aligned} C_\sigma + \frac{1}{2}V_\sigma &\geq C_\sigma + \frac{\mu}{36}|S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma|^{(4-\beta)/(2-\beta)}C_\sigma^{-2/(2-\beta)} \\ &\geq \check{\mu}S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma, \end{aligned}$$

where

$$\check{\mu} = \left|\frac{\mu}{18(2-\beta)}\right|^{(2-\beta)/(4-\beta)} + \frac{\mu}{36}\left|\frac{\mu}{18(2-\beta)}\right|^{-2/(4-\beta)}.$$

Therefore,

$$\lim_{n\to\infty}\{1 - \mathbb{E}[\exp\{-\alpha_n(-Z[X^n]_\sigma - C[S^n,\beta;X^n]_\sigma)\}]\} \leq 1 - \mathbb{E}[\exp\{\check{\mu}S^{2/(4-\beta)} \cdot \langle Y\rangle_\sigma\}].$$

The upper bound is asymptotically attained by the efficient scheme $X^n = X^n(\gamma)$ defined by (21) and (22) when $|\gamma| \to \infty$, where $\epsilon_n = \check{\nu}\alpha_n^{-1}$ and

$$\check{\nu} = \mu^{1/2}\left|\frac{\mu}{18(2-\beta)}\right|^{-1/(4-\beta)}\left|\frac{\cosh((\beta-1)\gamma)}{\cosh(\gamma)}\right|^{1/(2-\beta)}.$$

Consequently, the efficient schemes obtained in the preceding sections are in fact maximizers of the exponential utility in an asymptotic sense.

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